Hausner's parametric technique

A few weeks ago I stumbled over an intriguing technique described by Arthur Hausner[1] in [Hausner 1962]. In analog/hybrid computer programming differential equations such as Legendre's[2]

$$(1-x^2)y''(x) - 2xy'(x) - n(n+1)y(x) = 0$$

are a challenge since the naïve approach results in a fraction on the right hand side of

$$y''(x) = \frac{2xy'(x) - n(n+1)y(x)}{1-x^2}$$

which may be problematic for $x \to 1$.[3]

Hausner's technique is based on the chain rule: Given some DEQ of the form

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = y^{(n)} = \frac{p(x, y, y', \ldots, y^{(n-1)})}{q(x, y, y', \ldots, y^{(n-1)})}$$

this can be transformed into a set of DEQs of first order by defining y_i as $y_i = y^{(i)}$, i. e., $\frac{\mathrm{d}y}{\mathrm{d}x} = y_1$ and $\frac{\mathrm{d}y_i}{\mathrm{d}x} = y_{i+1}$ for $1 < i < n-1$ and thus

$$\frac{\mathrm{d}y_{n-1}}{\mathrm{d}x} = y_n = \frac{p(x, y, y', \ldots, y^{(n-1)})}{q(x, y, y', \ldots, y^{(n-1)})}.$$

Introducing a new parameter τ and differentiating with respect to it yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\tau}}{\frac{\mathrm{d}x}{\mathrm{d}\tau}} = \frac{\dot{y}}{\dot{x}} = y_1,$$

i. e., $\dot{y} = y_1\dot{x}, \ldots, \dot{y}_i = y_{i+1}\dot{x}$.

[1] 27.11.1929–15.10.2020, see https://www.santacruzsentinel.com/obituaries/arthur-hausner/, retrieved 03.08.2026

[2] Adrien-Marie Legendre, 18.09.1752–09.01.1833

[3] One way of solving this has been shown in application note #38: https://analogparadigm.com/downloads/alpaca_38.pdf, retrieved 03.08.2026.

This technique can now be applied to LEGENDRE's DEQ as an example. We have

$$y'' = \dot{y}_1 = \frac{p(x, y, y', \ldots, y^{(n-1)})}{q(x, y, y', \ldots, y^{(n-1)})}\dot{x} = \frac{2xy_1 - \eta y}{1 - x^2}\dot{x}$$

with $\eta := n(n+1), n \in \mathbb{N}$. Now the "nasty" denominator $1 - x^2$ can be eliminated by setting $\dot{x} := 1 - x^2$. This results in the following set of three coupled DEQs:

$$\begin{aligned}\dot{x} &= 1 - x^2\\ \dot{y} &= y_1\dot{x}\\ \dot{y_1} &= 2xy_1 - \eta y\end{aligned}$$

Voilà – the fraction is completely gone! Of course, there is no such thing like a free lunch – now three multipliers are required to implement this system. The only thing still left necessary is to scale it. x and y are no problem as long as x is restricted to, e. g., $[0, 1]$. However, y_1 need scaling. Assuming $n \leq 5$, $\max(\eta) = 30$, suggesting a scaling factor of $\lambda_{y_1} = 10^{-2}$:

$$\dot{x} = 1 - x^2 \tag{1}$$

$$\dot{y} = 10\widehat{y_1}\dot{x} \tag{2}$$

$$\dot{\widehat{y_1}} = 2xy_1 - \widehat{\eta}y \tag{3}$$

with $\widehat{\eta} = \frac{\eta}{100}$ and $k_{0_x} = k_{0_{y_1}} = 10^2$ and $k_{0_y} = 10^3$. This last k_{0_y} together with the additional weight 10 in the second equation compensates for λ_{y_1}. The resulting analog computer setup is shown in figure 1.

To generate the various Legendre polynomials as solution for this set of coupled DEQs, there are three parameters: η, $y(0)$, and $y_1(0)$ as shown in table 1. Due to the above scaling factor of $\lambda_{y_1} = 10^2$ the actual parameters are $\widehat{\eta} = \frac{\eta}{100}$ and $\widehat{y_1}(0) = \frac{y_1(0)}{100}$.

Due to the transformation $\dot{x} = 1 - x^2$ the resulting function can be displayed as a phase space plot by plotting x against y.[4] Figure 2 shows the result for $n = 5$, i. e., $\widehat{\eta} = 0.3$, $y(0) = 0$, and $\widehat{y_1}(0) = -0.0188$.

[4] This might require additional inverters.

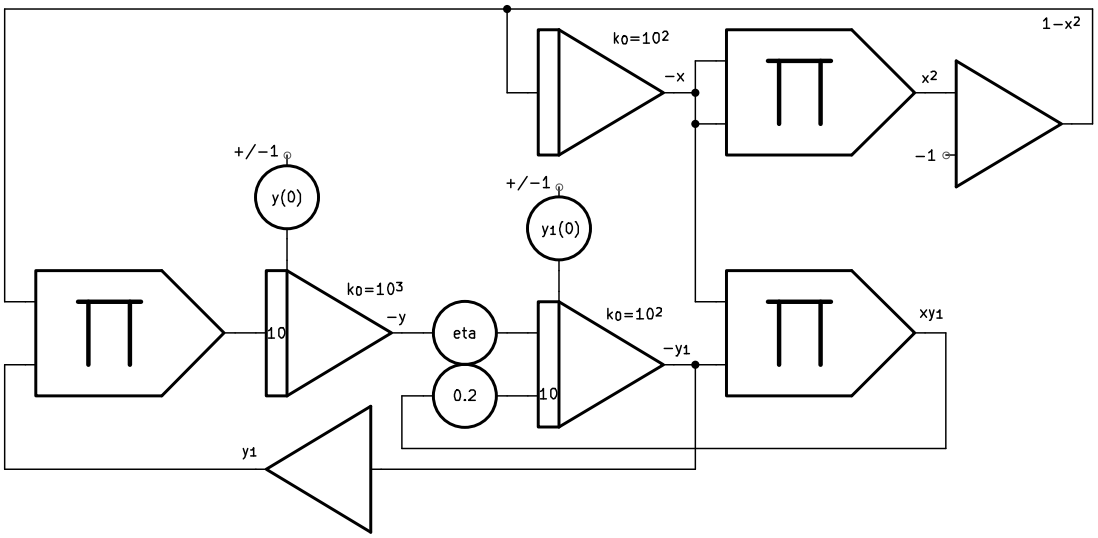

Figure 1: Analog computer program for equations (1), (2), and (3)

n	$P_n(t)$	$y(0)$	$\dot{y}(0)$
1	t	0	1
2	$\frac{3}{2}x^2 - \frac{1}{2}$	$-\frac{1}{2}$	0
3	$\frac{5}{2}x^3 - \frac{3}{2}x$	0	$-\frac{3}{2}$
4	$\frac{35}{8}x^4 - \frac{15}{4}x^3 + \frac{3}{8}$	$\frac{3}{8}$	0
5	$\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$	0	$\frac{15}{8}$

Table 1: The first five $P_n(t)$ solving LEGENDRE's DEQ with the resulting initial conditions

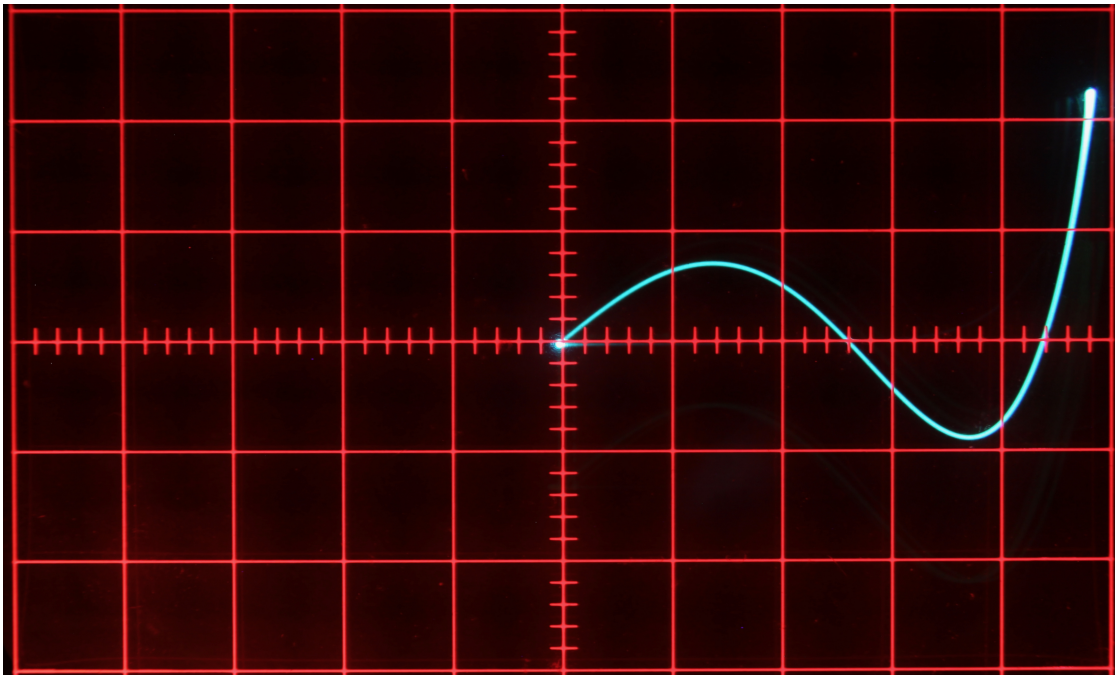

Figure 2: Solution for the LEGENDRE DEQ for $n = 5$

References

[Hausner 1962] Arthur Hausner, "Parametric Techniques for Eliminating Division and Treating Singularities in Computer Solutions of Ordinary Differential Equations", in *IRE Transactions on Electronic Computers*, February 1962, pp. 42–45