Analog Computer Applications
Application Note #8
Undergraduate Engineering Physics Space-Themed STEM Project

Rocket Propulsion: Lunar Touchdown
(A Simulation)

1 Introduction

For this application note, a hypothetical space mission is designed to deliver a delicate seismic sensor to the surface of the moon. To accomplish this, a probe, consisting of a two-stage rocket (a parachute will not work!) is used to ensure that the sensor at touchdown will arrive unscathed, with a speed of 0 m/s. During Phase I, the probe descends from an initial orbital speed to a speed of -3 m/s as it reaches an altitude of 16 meters. At that point, the second-stage rocket is released with its engine activated for the final 7 seconds of descent (Phase II). Just as the probe touches down on the lunar surface, the engine must be disengaged to prevent lift off. This simulation will focus only on Phase II of the probe's final 7 seconds of descent.

For comparison, analytical, numerical method, and analog computation solutions of the probe's speed will be determined and displayed in table 1.

2 Mathematical modeling

Starting with the basic rocket equation (derived in Application Note #9),

$$\lim_{\Delta t \to 0} (F_{ext}) = m dv/dt + u dm/dt, \text{ where}$$

$\lim_{\Delta t \to 0} (F_{ext})$ = -weight of rocket, including fuel = -mg. Thus,

$$-mg = m dv/dt + u dm/dt$$

$$dv/dt = -g - u/m \, dm/dt$$

Letting dm/dt = constant = -c, where = c > 0 and $m(0) = m_0$,

$$m = -ct + m_0$$

Assuming $m_0 = 0.900$ kg, c = 0.075 kg/s (exhaust ejection rate), u = 17.2 m/s (engine exhaust speed relative to engine), g = 1.625 m/s^2 (lunar surface acceleration), and omitting units for simplicity,

$$dv/dt = -1.625 + 1.290/(-0.075t + 0.900) \text{ with } v(0) = -3.000 \text{ m/s} \qquad (1)$$

Equation (1) is used for computer setup.

Now, for an analytical solution (fractions used for convenience):

Multiplying the numerator and denominator of the fraction on the right by -40/3 and replacing 1.625 with 13/8,

$$dv/dt = -13/8 - (86/5)1/(t - 12)$$

$$dv = -13/8\, dt - (86/5)\, dt/(t - 12)$$

$$\int_{-3}^{v} dv = -13/8 \int_{0}^{t} dt - 86/5 \int_{0}^{t} dt/(t - 12) \qquad (2)$$

Integrating (2) by inspection,

$$v = -3 - 13/8\, t - 86/5 \ln(1 - t/12) \qquad (3)$$

For this simulation, the computer setup is not designed to determine the probe's distance from the point of ignition to the lunar surface, with s(0) = 0. Nonetheless, an analytical solution will be developed.

Letting v = ds/dt,

$$ds/dt = -3 - 13/8\, t - 86/5 \ln(1 - t/12)$$

$$ds = -3dt - 13/8\, tdt - 86/5 \ln(1 - t/12)\, dt$$

$$\int_0^s ds = -3\int_0^t dt - 13/8\int_0^t tdt - 86/5\int_0^t \ln\left(1-\frac{t}{12}\right)dt \qquad (4)$$

Integration details for evaluating

$$\int \ln\left(1-\frac{t}{12}\right)dt$$

Letting p = (1 – t/12), then dt = -12dp.

$$\int \ln\left(1-\frac{1}{12}\right)dt = -12\int \ln(p)dp$$

Letting u = ln(p) and dv = dp, then du = dp/p and v = p.

Integrating by parts,

$$\int udv = uv - \int vdu$$

$$\int \ln(p)dp = \ln(p)p - \int pdp/p$$

$$\int \ln(p)dp = p\ln(p) - \int dp$$

$$\int \ln(p)dp = p(\ln(p) - 1)$$

Yielding,

$$\int_0^t \ln\left(1-\frac{t}{12}\right)dt = -12[\left(\ln\left(1-\frac{t}{12}\right)-1\right)+1]$$

Integrating (4) in part by inspection and using the above result,

$$s = -3t - 13/16\, t^2 - 86/5(-12)[(1 - t/12)(\ln(1 - t/12) - 1) + 1]$$

$$s = -3t - 13/16\, t^2 + 1032/5[(1 - t/12)(\ln(1 - t/12) - 1) + 1] \qquad (5)$$

Setting v = 0 in (3) and solving for touchdown time (t'),

$$0 = -3 - 13/8\, t' - 86/5 \ln(1 - t'/12)$$

Using graphing techniques and inserting units, t' = 6.594 ≅ 7 seconds.

Substituting 6.594 into (5) and inserting units, s = -15.838 ≅ -16 meters.

3 Computer setup

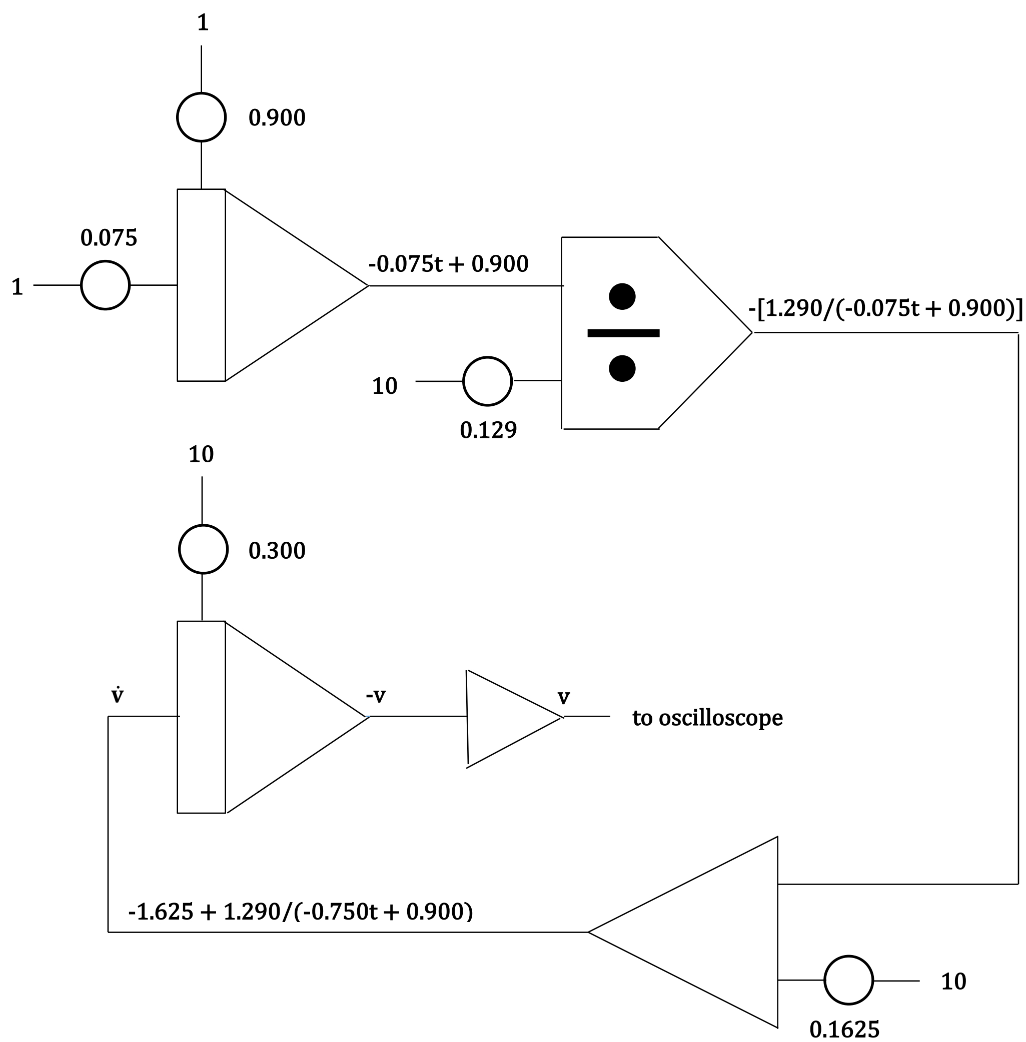

Figure 1: Computer setup for Rocket Propulsion: Lunar Touchdown

4 Numerical Method

(Modified Euler method using a hand-held programmable calculator)

```
PROGRAM:TOUCHDWN
:ClrHome:ClrDraw
:"DV/DT= -1.625+1.290/(-0.075t+0.900)"
:"INITIAL CONDITION:"
:0→T:-3→V
:"STEP SIZE:"
:0.100→H
:Fix 2
:Lbl 1
:If T>7:Then
:Goto 2:Else
:Disp {T,V}
:-1.625+1.290/(-0.075t+0.900)→F
:T+H→T
:-1.625+1.290/(-0.075t+0.900)→S
:(F+S)/2→A
:V+AH→V
:Pause
:Goto1
:Lbl 2
:End
```

5 Results

Oscilloscope Time t(s)	engine status	Analog Computation v(m/s)	Analytical v(m/s)	Numerical Method v(m/s)
0.0	engaged	-3.0	-3.00	-3.00
1.0	engaged	-3.0	-3.13	-3.13
2.0	engaged	-3.0	-3.11	-3.11
3.0	engaged	-2.9	-2.93	-2.93
4.0	engaged	-2.5	-2.53	-2.53
5.0	engaged	-1.8	-1.85	-1.85
6.0	engaged	-0.9	-0.83	-0.83
6.6	disengaged	0.0	0.00	0.01
7.0	if not disengaged	0.6	0.68	0.68

Table 1: Solution Comparisons

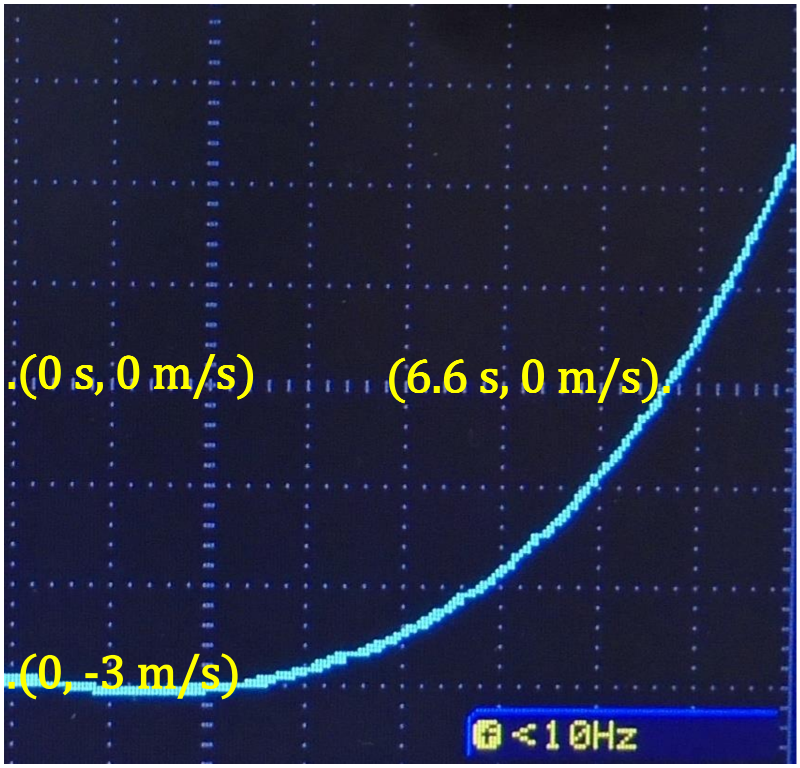

Figure 2: Probe's speed as a function of time (- indicates downward)

*For this application note, the display was produced during a single run by a differential equation analog computer prototype using discrete components with tolerances between 1% and 10%.