Analog Computer Applications
Application Note #9
Undergraduate Engineering Physics STEM Project

Newton's Law of Cooling: An Introduction to Scaling

1 Introduction

This simulation applies scaling to solve a relatively simple differential equation involving Newton's law of cooling. Initially at 100°C, a hot cup of tea is placed in a room that is maintained at an ambient constant temperature of 20°C.

For comparison, analytical, numerical, and analog computer solutions are displayed in table 1 on page 6.

Please note: Originally, the rocket equation derivation was going to be part of Application Note #9. Instead. it will be included in Application Note #10. Sorry for any inconvenience this may have caused.

2 Mathematical modeling

Starting with Newton's law of cooling,

$$dT/dt = -k(T - T_A) \text{ with } T(0) = T_0, \text{ where} \quad (1)$$

T = temperature of the tea
t = time
k = temperature coefficient
T_A = ambient temperature (assumed constant)
T_0 = initial temperature of the tea

Letting $k = 0.25 \text{ min}^{-1}$, $T_A = 20^oC$, and $T(0) = 100^oC$,

$$dT/dt = -0.25\text{min}^{-1}(T - 20^oC) \text{ with } T(0) = 100^oC \quad (2)$$

Following a bit of calculus (details will be provided for the scaled version),

$$T(t) = 80^{o}Ce\text{^}(-0.25\ min^{-1}t) + 20^{o}C \quad (3)$$

As a reminder, operational amplifier (op amp) input/output voltages are between V_{EE} and V_{CC}. For this project, 9-Volt batteries were used (V_{EE} = -9 Volts and V_{CC} = +9 Volts). Allowing for a safety of margin, to prevent saturation, voltages are kept between -6 Volts and +6 Volts.

The issue of magnitudes is obvious. A value like 100°C is just too big for direct conversion to 100 Volts. So, a temperature of 100°C will be reduced (scaled down) to an analog voltage of 6 Volts and 20°C will be reduced to an analog voltage of 1.20 Volts.

Time will be scaled such that 1 minute = 1 second. Easy enough!

To start, let

$R \equiv$ Reducing scale factor = T_{max}/V_{max} = 100°C/6 Volts = 50°C/3 Volts.

In general, R = T/V or T = RV.

Replacing T with RV, (1) becomes

$$d(RV)/dt = -k(RV - T_A) \text{ with } V(0) = T(0)/R = T_o/R$$

$$RdV/dt = -kR(V - T_A/R) \text{ with } V(0) = T(0)/R = T_o/R$$

$$dV/dt = -k(V - V_A) \text{ with } V_A = T_A/R \text{ and } V(0) = T(0)/R = T_o/R$$

Inserting values, but omitting units for clarity,

$$dV/dt = -0.25(V - 20/(50/3)),\ \ V(0) = 100/(50/3)$$

$$dV/dt = -0.25(V - 1.20) \text{ with } V(0) = 6.00 \quad (4)$$

$$dV/(V - 1.20) = -0.25\ dt$$

$$\int_{6}^{T} \frac{dV}{V - 1.20} = -0.25 \int_{0}^{t} dt$$

Integrating by inspection, and noting that $V > 1.20$,

$$\ln((V - 1.20)/(6 - 1.20)) = -0.25t$$

$$\ln((V - 1.20)/4.8) = -0.25t$$

$$(V - 1.20)/4.8 = e^{(-0.25t)}$$

$$V = 4.80\, e^{(-0.25t)} + 1.20$$

Inserting units,

$$V(t) = 4.80 \text{ Volts } e^{(-0.25 \text{ min}^{-1} t)} + 1.20 \text{ Volts} \qquad (5)$$

Since $T = RV$,

$$T(t) = 50^{o}/3 \text{ Volts} \times [4.80 \text{ Volts } e^{(-0.25 \text{ min}^{-1} t)} + 1.20 \text{ Volts}]$$

$$T(t) = 80^{o}C\, e^{(-0.25 \text{ min}^{-1} t)} + 20^{o}C, \text{ which is identical to (3)}$$

3a Computer setup (scaled, patch cord version)

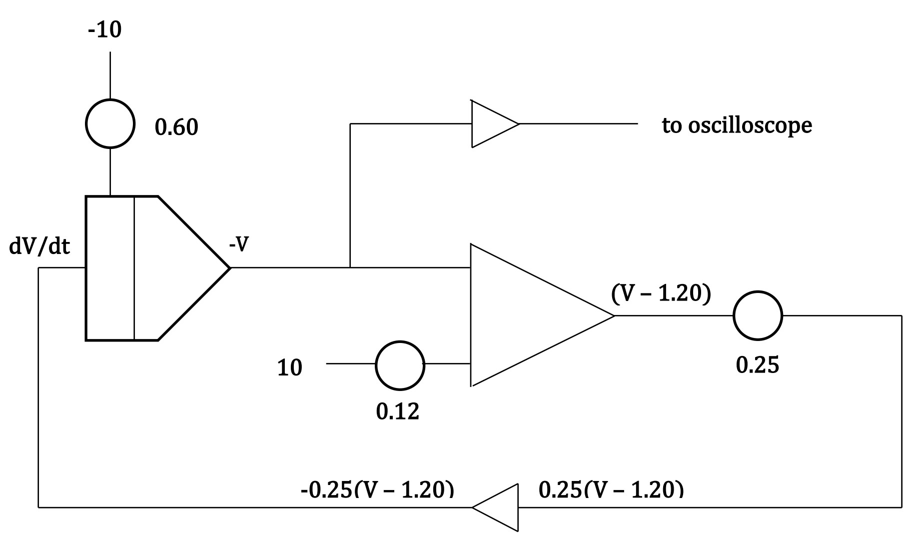

Figure 1: Computer setup for Newton's law of cooling

3b Computer setup (op amp/discrete component version)

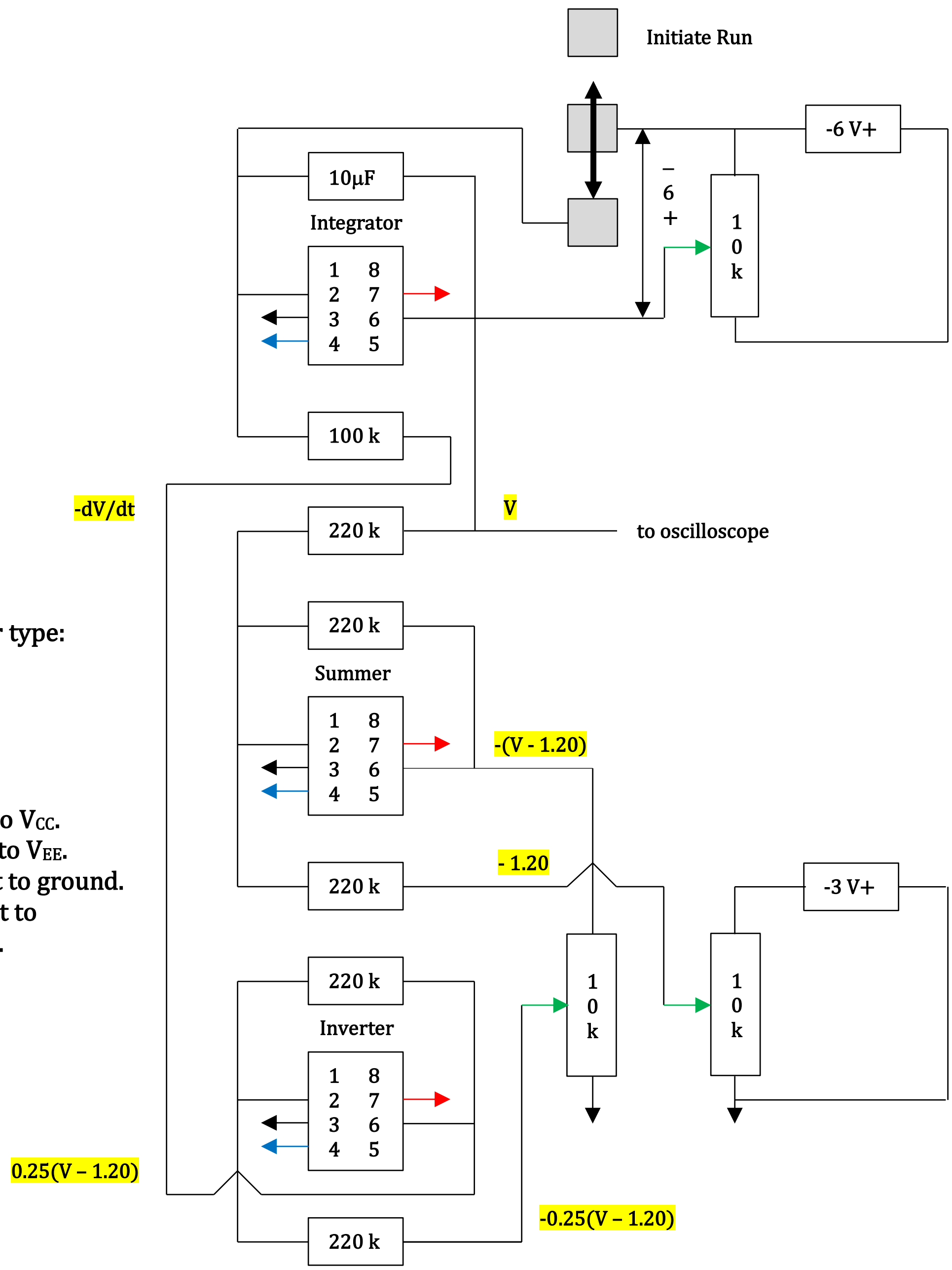

Figure 2: Basic breadboard layout

Operational amplifier type:

μA741CN

Arrow Color Key:

Red arrows connect to V_{CC}.
Blue arrows connect to V_{EE}.
Black arrows connect to ground.
Green arrows connect to component indicated.

4 Numerical Method

(Modified Euler method using a hand-held programmable calculator)

Code: TI-BASIC

```
PROGRAM:COOLING
:ClrHome:ClrDraw
:"NEWTONS LAW"
:"OF COOLING"
:"DV/DT=-0.25(V–1.20)"
:"WITH V(0)=6.00"
:"V=VOLTAGE ANALOG"
:"OF TEMPERATURE"
:"T =TIME"
:"PARAMETERS:"
:0→T:6.00→V:0.25→H
:"H=STEP SIZE"
: Fix 1
:Lbl 1
:If T>20:Then
:Goto 2: Else
:DiSP {T,V}
:-0.25(V–1.20)→F
:V+HF→W
:T+H→T
: -0.25(W–1.20)→S
:(F+S)/2→A
:V+AH→V
:Pause
:Goto1
:Lbl 2
:End
```

5 Results

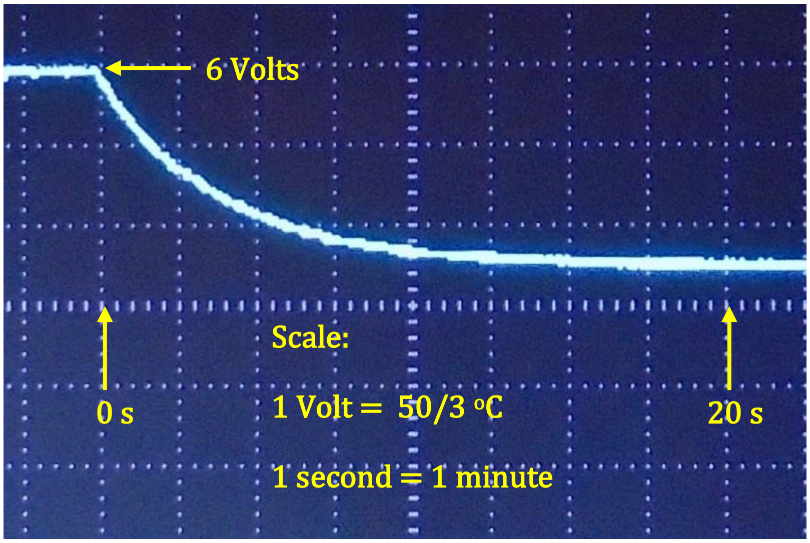

Figure 3: Voltage vs time simulation*

*For this application note, the oscilloscope display was produced during a single run using a differential equation analog computer constructed from operational amplifiers and discrete components with tolerances within 10%.

t (s) scaled from minutes	Analog Computer V (Volts) Estimated from oscilloscope	Analog Computer Converted T (°C)	Analytical T (°C)	Numerical T (°C)
00.0	6.0	100	100	100
02.5	3.6	60	63	63
05.0	2.6	43	43	43
07.5	2.0	33	32	32
10.0	1.6	27	27	27
12.5	1.3	22	24	24
15.0	1.2	20	22	22
17.5	1.2	20	21	21
20.0	1.2	20	21	21

Table 1: Solution Comparisons

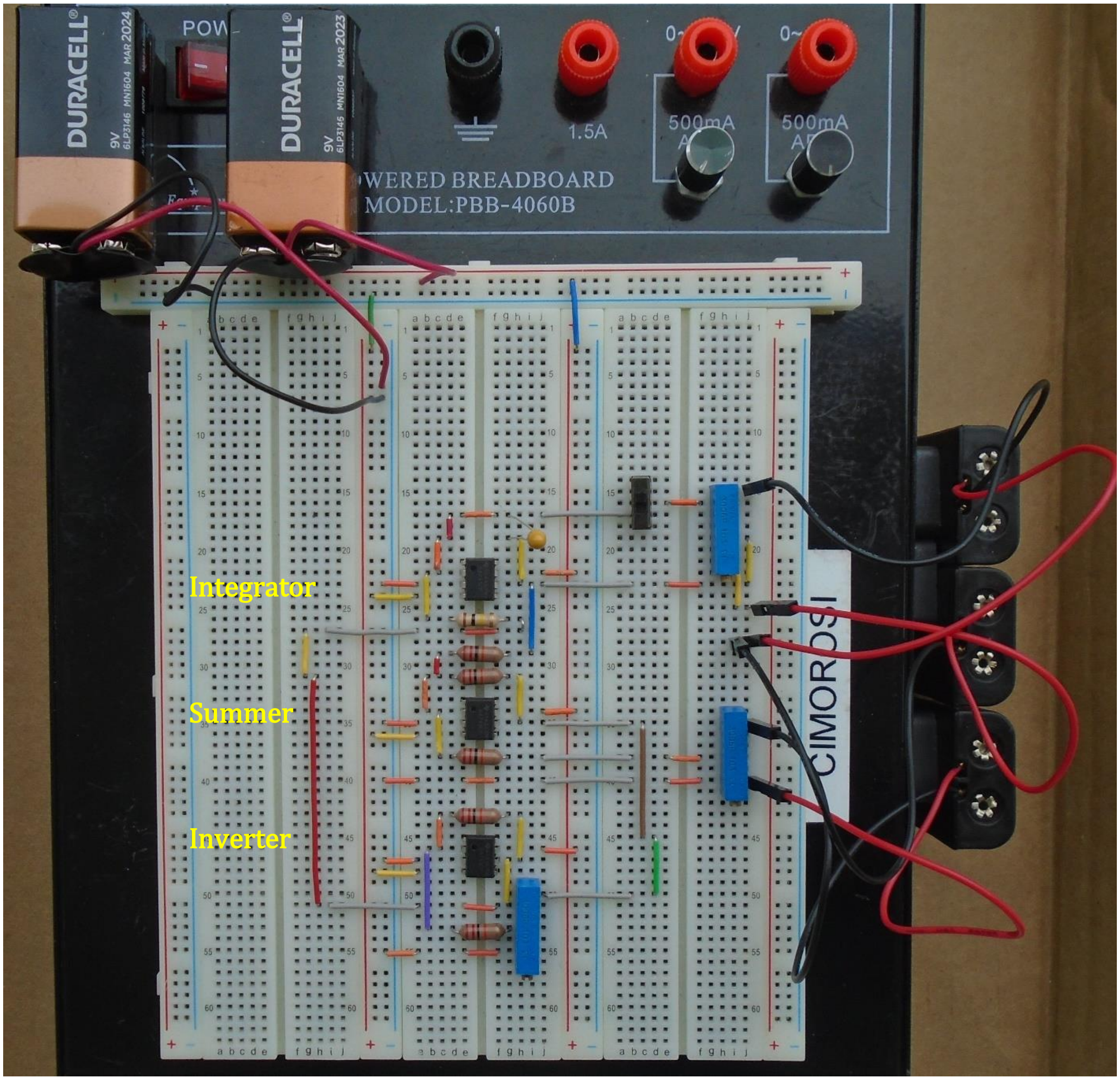

Figure 4: Differential Equation Analog Computer

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